

UNIQUE DECOMPOSITIONS, FACES, AND AUTOMORPHISMS OF SEPARABLE STATES

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ABSTRACT. Let S_k be the set of separable states on $\mathcal{B}(\mathbb{C}^m \otimes \mathbb{C}^n)$ admitting a representation as a convex combination of k pure product states, or fewer. If $m > 1, n > 1$, and $k \leq \max(m, n)$, we show that S_k admits a subset V_k such that V_k is dense and open in S_k , and such that each state in V_k has a unique decomposition as a convex combination of pure product states, and we describe all possible convex decompositions for a set of separable states that properly contains V_k . In both cases we describe the associated faces of the space of separable states, which in the first case are simplexes, and in the second case are direct convex sums of faces that are isomorphic to state spaces of full matrix algebras. As an application of these results, we characterize all affine automorphisms of the convex set of separable states, and all automorphisms of the state space of $\mathcal{B}(\mathbb{C}^m \otimes \mathbb{C}^n)$ that preserve entanglement and separability.

1. INTRODUCTION

A state on the algebra $\mathcal{B}(\mathbb{C}^m \otimes \mathbb{C}^n)$ of linear operators is separable if it is a convex combination of product states. States that are not separable are said to be entangled, and are of substantial interest in quantum information theory. Easily applied conditions for separability are known only for special cases, e.g., if $m = n = 2$, then a state is separable iff its associated density matrix has positive partial transpose, cf. [14, 6]. Other necessary and sufficient conditions are known, e.g. [6], but are not easily applied in practice. An open question of great interest is to find a simple necessary and sufficient condition for a state to be separable.

A product state $\omega \otimes \tau$ is a pure state iff ω and τ are pure states. Thus a separable state is precisely one that admits a representation as

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a convex combination of pure product states. It is natural to ask the extent to which this decomposition is unique. That is the main topic of this article.

For the full state space K of $\mathcal{B}(\mathbb{C}^m \otimes \mathbb{C}^n)$ each non-extreme point can be decomposed into extreme points in many different ways. But for the space S of separable states the situation is totally different. While non-extreme points with many different decompositions exist (and are easy to find) in S as well as in K , there are in S also plenty of points for which the decomposition is unique.

DiVincenzo, Terhal, and Thapliyal [4] defined the *optimal ensemble cardinality* of a separable state ρ to be k if k is the minimal number of pure product states required for a convex decomposition of ρ . Lockhart [11] used the term “optimal ensemble length” for the same notion. For brevity, we will simply call this number the *length* of ρ , and we denote the set of separable states of length at most k by S_k . We show in Theorem 6 that for $m > 1, n > 1$ and $k \leq \max(m, n)$, the set S_k has a subset V_k which is dense and open in S_k , with each $\sigma \in V_k$ admitting a unique decomposition into pure product states. Actually, we exhibit such a set V_k consisting of states with the property that each generates a face of S which is a simplex, from which the uniqueness follows.

We remark that the sets V_k are open and dense in the relative topology on S_k , but are not open or dense in S or K if $mn > 1$. (See the remarks after Theorem 6). Indeed it would be surprising if a subset of low rank separable states were open and dense in the set of all states of that rank, since low rank states are almost surely entangled [17, 22], and in general S has measure which is a decreasingly small fraction of the measure of K as m, n increase, cf. [3, 20].

While dimensions are too high to be able to accurately visualize the above results, the reader may be curious about the relationship to the well known tetrahedron/octahedron picture for $m = n = 2$, cf. [5]. In that picture, there is a subset \mathcal{T} of states which is a tetrahedron, and which has the property that for every state ρ which restricts to the normalized trace on $\mathcal{B}(\mathbb{C}^2) \otimes I$ and on $I \otimes \mathcal{B}(\mathbb{C}^2)$, there are unitaries U and V such that $(U \otimes V)^* \rho (U \otimes V) \in \mathcal{T}$. The midpoints of the six edges of this tetrahedron are the vertices of an octahedron that consists of the separable states in \mathcal{T} . Each vertex of the octahedron is a convex combination of two distinct pure product states (which of course are not in \mathcal{T}), cf. [12, eqn. (63)]. In fact, the vertices are the only states in the octahedron of length $\leq \max(m, n) = 2$.

It can be checked (e.g., by applying our Corollary 5) that the decomposition of each of these vertices into pure product states is unique. Each state in the interior of this octahedron has rank $4 = mn$, so is an

interior point of the full state space K , hence has a non-unique convex decomposition into pure product states (see the remarks after Theorem 6.) The tetrahedron also arises as a parameterization for a set of unital completely positive trace preserving maps from $M_2(\mathbb{C})$ to $M_2(\mathbb{C})$, with the octahedron consisting of the entanglement breaking maps in this set, cf. [16, Appendix B], [15, Thm. 4], and [17, Fig. 2].

We also define a broader class of states that we show have a unique decomposition as a convex combination of product states $\rho_i \otimes \sigma_i$ that are not necessarily pure, but with the property that each of them generates a face of S which is also a face of K and is affinely isomorphic to the state space of $\mathcal{B}(\mathbb{C}^{p_i})$ for a suitable p_i . From this it follows that the ambiguity in decompositions for a given state in this class is restricted to the ambiguity in decompositions for points in the state space of the matrix algebras $\mathcal{B}(\mathbb{C}^{p_i})$. For a complete description of the possible decompositions of a state on $\mathcal{B}(\mathbb{C}^p)$, see [10, 18, 24].

We use our results on the facial structure of S to show that every affine automorphism of the space S of separable states on $\mathcal{B}(\mathbb{C}^m \otimes \mathbb{C}^n)$ is given by a composition of the duals of the maps that are (i) conjugation by local unitaries (i.e., unitaries of the form $U_1 \otimes U_2$) (ii) the two partial transpose maps, or (iii) the swap automorphism that takes $A \otimes B$ to $B \otimes A$ (if $m = n$). A consequence is a description of the affine automorphisms Φ of the state space such that Φ preserves entanglement and separability.

There is related work of Hulpke et al [7]. They say a linear map L on $\mathbb{C}^m \otimes \mathbb{C}^n$ preserves *qualitative entanglement* if L sends separable (i.e., product) vectors to product vectors, and entangled vectors to entangled vectors. They show that a linear map L preserves qualitative entanglement of vectors on $\mathbb{C}^m \otimes \mathbb{C}^n$ iff L is a local operator (i.e. one of the form $L_1 \otimes L_2$), or if L is a local operator composed with the swap map that takes $x \otimes y$ to $y \otimes x$. They then show that if L preserves a certain *quantitative* measure of entanglement, then L must be a local unitary.

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2. BACKGROUND: STATES ON $\mathcal{B}(\mathbb{C}^n)$

We review basic facts about states on $\mathcal{B}(\mathbb{C}^n)$, and develop some facts about the relationship of independence of vectors x in \mathbb{C}^n and of the corresponding vector states ω_x . In the following sections we will specialize to the case of interest: separable states.

Notation. If x is a vector in any vector space, $[x]$ denotes the subspace generated by x . \mathbb{C}^n denotes the set of n -tuples of complex numbers

viewed as an inner product space with the usual inner product (linear in the first factor). $\mathcal{B}(\mathbb{C}^n)$ denotes the linear transformations from \mathbb{C}^n into itself. For each unit vector $x \in \mathbb{C}^n$, we denote the associated vector state by ω_x , so that $\omega_x(A) = (Ax, x)$. The convex set of states on $\mathcal{B}(\mathbb{C}^n)$ will be denoted by K_n .

We recall that faces of the state space K_n of $\mathcal{B}(\mathbb{C}^n)$ are in 1-1 correspondence with the projections in $\mathcal{B}(\mathbb{C}^n)$, and thus with the subspaces of \mathbb{C}^n that are the ranges of these projections. If Q is a projection in $\mathcal{B}(\mathbb{C}^n)$, then the associated face F_Q of K_n consists of all states taking the value 1 on Q . The restriction map is an affine isomorphism from F_Q onto the state space of $Q\mathcal{B}(\mathbb{C}^n)Q \cong \mathcal{B}(Q(\mathbb{C}^n))$. Thus F_Q is affinely isomorphic to the state space of $\mathcal{B}(L)$, where $L = Q(\mathbb{C}^n)$. The set of extreme points of K_n are the vector states, and it follows that the extreme points of F_Q are the vector states ω_x with x in the range of Q , and F_Q is the convex hull of these vector states. For background, see [2, Chapter 4]

Definition. Recall that a convex set C is said to be the *direct convex sum* of a collection of convex subsets C_1, \dots, C_p if each point $\omega \in C$ can be uniquely expressed as a convex combination

$$(1) \quad \omega = \sum_{i \in I} \lambda_i \omega_i$$

where $I \subset \{1, \dots, p\}$, $\lambda_i > 0$ for all $i \in I$, $\omega_i \in C_i$ for all $i \in I$, and $\sum_{i \in I} \lambda_i = 1$.

If C is a convex subset of a real linear space and is located on an affine hyperplane which does not contain the origin (as is the case for our state spaces), then it is easily seen that C is the direct convex sum of convex subsets C_1, \dots, C_p iff the span of C is the direct sum of the real subspaces spanned by C_1, \dots, C_p .

A finite dimensional convex set is a *simplex* if it is the direct convex sum of a finite set of points. If the affine span of the points does not contain the origin, then their convex hull is a simplex iff the points are linearly independent (over \mathbb{R}).

Lemma 1. *Let L be a subspace of \mathbb{C}^n and suppose that L is the direct sum of subspaces L_1, \dots, L_p . Let F_1, \dots, F_p be the corresponding faces of the state space of $\mathcal{B}(\mathbb{C}^n)$. Then the convex hull of F_1, \dots, F_p is the direct convex sum of those faces. In particular, if x_1, \dots, x_p are linearly independent unit vectors, then the corresponding vector states are linearly independent and the convex hull of the corresponding vector states is a simplex.*

Proof. Let $I \subset \{0, \dots, p\}$, and suppose $\{\omega_i \mid i \in I\}$ are nonzero functionals on $\mathcal{B}(\mathbb{C}^n)$ with $\omega_i \in \text{span}_{\mathbb{R}} F_i$ for each i . To prove independence of $\{\omega_i \mid i \in I\}$, suppose that for scalars $\{\gamma_i\}_{i \in I}$ we have

$$(2) \quad \sum_{i \in I} \gamma_i \omega_i = 0.$$

Let L_0 be the orthogonal complement of L . Then \mathbb{C}^n as a linear space is the direct sum of L_0, L_1, \dots, L_p .

For each $i \in I$, let P_i be the projection associated with F_i . Then we can find $A_i \in P_i \mathcal{B}(\mathbb{C}^n) P_i$ such that $\omega_i(A_i) \neq 0$. Let $B_i \in \mathcal{B}(\mathbb{C}^n)$ be an operator such that B_i is zero on $\sum_{j \neq i} L_j$, and such that $\omega_i(B_i) \neq 0$ (e.g., set $B_i = A_i$ on L_i). If $x \in L_j$ and $j \neq i$, then $\omega_x(B_i) = (B_i x, x) = 0$. Since every state in F_j is a convex combination of vector states ω_x with $x \in L_j$, then $\omega_j(B_i) = 0$ if $j \neq i$.

Now apply both sides of (2) to B_k to conclude that $\gamma_k \omega_k(B_k) = 0$ for all $k \in I$, so $\gamma_k = 0$ for all $k \in I$. Thus the set of vectors $\omega_1, \dots, \omega_p$ is independent. We conclude that $\text{co}(F_1, \dots, F_p)$ is the direct convex sum of F_1, \dots, F_p .

If x_1, \dots, x_p are linearly independent unit vectors, applying the result above with $F_i = \{\omega_{x_i}\}$ shows that the convex hull of the vector states ω_{x_i} is a simplex. Hence the set $\{\omega_{x_1}, \dots, \omega_{x_p}\}$ is linearly independent. \square

Note that the converses of the statements above are not true. For example, while no set of more than two vectors in \mathbb{C}^2 is independent, it is easy to find a set of three linearly independent vector states on $\mathcal{B}(\mathbb{C}^2)$.

3. UNIQUENESS OF DECOMPOSITIONS OF SEPARABLE STATES

We now turn to faces of the set of separable states on $\mathcal{B}(\mathbb{C}^m \otimes \mathbb{C}^n)$, and to the question of uniqueness of convex decompositions of such states. We identify $\mathcal{B}(\mathbb{C}^m \otimes \mathbb{C}^n)$ with $\mathcal{B}(\mathbb{C}^m) \otimes \mathcal{B}(\mathbb{C}^n)$ by $(A \otimes B)(x \otimes y) = Ax \otimes By$. We denote the convex set of all states on $\mathcal{B}(\mathbb{C}^m \otimes \mathbb{C}^n)$ by K , and the convex set of all separable states by S .

Lemma 2. *Let e_1, e_2, \dots, e_p and f_1, f_2, \dots, f_p be unit vectors in \mathbb{C}^m and \mathbb{C}^n respectively. We assume that f_1, f_2, \dots, f_p are linearly independent. If $e \in \mathbb{C}^m$ and $f \in \mathbb{C}^n$ are unit vectors such that $e \otimes f$ is in the linear span of $\{e_i \otimes f_i \mid 1 \leq i \leq p\}$, then there is an index j such that $[e] = [e_j]$ and such that f is in the span of those f_i such that $[e_i] = [e_j]$. In the special case where $[e_1], \dots, [e_p]$ are distinct, then $[e] = [e_j]$ and $[f] = [f_j]$ for some index j , and $\{e_i \otimes f_i \mid 1 \leq i \leq p\}$ is independent.*

Proof. Extend f_1, \dots, f_p to a basis f_1, \dots, f_n of \mathbb{C}^n , and let $\widehat{f}_1, \dots, \widehat{f}_n$ be the dual basis. For $1 \leq k \leq n$, let $T_k : \mathbb{C}^m \otimes \mathbb{C}^n \rightarrow \mathbb{C}^m$ be the linear map such that $T_k(x \otimes y) = \widehat{f}_k(y)x$ for $x \in \mathbb{C}^m, y \in \mathbb{C}^n$.

Suppose that the product vector $e \otimes f$ is a linear combination

$$(3) \quad e \otimes f = \sum_{i=1}^p \alpha_i e_i \otimes f_i.$$

For $j > p$, applying T_j to both sides of (3) gives $\widehat{f}_j(f)e = 0$, so $\widehat{f}_j(f) = 0$ for all such j . Now if $1 \leq j \leq p$, applying T_j to both sides of (3) gives

$$(4) \quad \widehat{f}_j(f)e = \alpha_j e_j.$$

Since $\widehat{f}_j(f)$ can't be zero for all j , then e is a multiple of some e_j . Fix such an index j . If $1 \leq i \leq p$ and $[e_i] \neq [e_j]$, then e_i can't be a multiple of e , so $\widehat{f}_i(f)e = \alpha_i e_i$ implies $\alpha_i = 0$, and then also $\widehat{f}_i(f) = 0$. We have shown that $\widehat{f}_i(f) = 0$ if $i > p$, or if $i \leq p$ and $[e_i] \neq [e_j]$. It follows that f is in the linear span of those f_i such that $[e_i] = [e_j]$.

If it also happens that $[e_1], \dots, [e_p]$ are distinct, and $[e] = [e_j]$, then $[f] = [f_j]$. Suppose now that $\sum_i \alpha_i e_i \otimes f_i = 0$. If $\alpha_k \neq 0$, then $e_k \otimes f_k$ is a linear combination of $\{e_i \otimes f_i \mid i \neq k\}$. Thus by the conclusion just reached, we must have $[e_k] = [e_i]$ for some $i \neq k$, contrary to the hypothesis that $[e_1], \dots, [e_p]$ are distinct. We conclude that $\alpha_k = 0$ for all k , and we have shown that $\{e_i \otimes f_i \mid 1 \leq i \leq p\}$ is independent. \square

Lemma 3. *Let $e_1, \dots, e_p \in \mathbb{C}^m$ and $f_1, \dots, f_p \in \mathbb{C}^n$ be unit vectors. If $[e_1] = [e_2] = \dots = [e_p]$, then the face F of S generated by the states $\{\omega_{e_i \otimes f_i} \mid 1 \leq i \leq p\}$ is also a face of K , and this face of K is associated with the subspace $L = e_1 \otimes \text{span}\{f_1, \dots, f_p\}$ of $\mathbb{C}^m \otimes \mathbb{C}^n$, and F is affinely isomorphic to the state space of $\mathcal{B}(L)$.*

Proof. Let G be the face of K which is associated with the subspace L of $\mathbb{C}^m \otimes \mathbb{C}^n$. By assumption each e_i is a multiple of e_1 , so that

$$L = \text{span}\{e_1 \otimes f_i \mid 1 \leq i \leq p\} = \text{span}\{e_i \otimes f_i \mid 1 \leq i \leq p\}.$$

Hence G is the face of K generated by $\{\omega_{e_i \otimes f_i} \mid 1 \leq i \leq p\}$.

We would like to show $G = F$. For brevity we denote the convex hull of the set $\{\omega_{e_i \otimes f_i} \mid 1 \leq i \leq p\}$ by C , and observe that G and F are the faces of K and S respectively generated by C . It follows easily from the definition of a face that the face generated by the convex set C in either one of the two convex sets S or K consists of all points ρ in S or K respectively which satisfy an equation

$$(5) \quad \omega = \lambda \rho + (1 - \lambda) \sigma$$

where $0 < \lambda < 1$, $\omega \in C$, and where σ is in S or K respectively. It follows that $F = \text{face}_S(C) \subset \text{face}_K(C) = G$.

Since each vector in L is a product vector, the extreme points of G are pure product states, so $G \subset S$. If ρ is in the face G of K generated by C , then we can find $\sigma \in K$ and $\omega \in C$ such that (5) holds. Then σ is also in $G \subset S$, so both ρ and σ are in S . Hence ρ is in the face F of S generated by C . Thus $G \subset F$, and so $F = G$ follows. \square

So far we have considered collections of product vectors $\{e_i \otimes f_i\}$ with $\{f_1, \dots, f_p\}$ linearly independent. In Lemma 3 we have described the face F of S generated these states in the special case where all of the e_i are multiples of each other. In this case F is also a face of K .

We now remove the restriction that all of the one dimensional subspaces $[e_i]$ coincide. We are going to partition the set of vectors $e_i \otimes f_i$ into subsets for which these subspaces coincide, and apply Lemma 3 to each such subset. For simplicity of notation, we renumber the vectors in the fashion we now describe.

Theorem 4. *Let e_1, e_2, \dots, e_p and f_1, f_2, \dots, f_p be unit vectors in \mathbb{C}^m and \mathbb{C}^n respectively, and with f_1, \dots, f_p linearly independent. We assume that the vectors are ordered so that $[e_1], \dots, [e_q]$ are distinct, and so that for $i > q$ each $[e_i]$ equals one of $[e_1], \dots, [e_q]$. For $1 \leq i \leq q$, let F_i be the face of S generated by the states $\{\omega_{e_j \otimes f_j} \mid [e_j] = [e_i]\}$ and $1 \leq j \leq p\}$. Then each F_i is also a face of K , and the face F of S generated by $\{\omega_{e_i \otimes f_i} \mid 1 \leq i \leq p\}$ is the direct convex sum of F_1, \dots, F_q . Moreover, each F_i is affinely isomorphic to the state space of $\mathcal{B}(L_i)$, where $L_i = e_i \otimes \text{span}\{f_j \mid [e_i] = [e_j]\}$. In the special case when $[e_1], \dots, [e_p]$ are distinct, then F is the convex hull of $\{\omega_{e_i \otimes f_i} \mid 1 \leq i \leq p\}$, and F is a simplex.*

Proof. By Lemma 3, the face F_i of S is equal to the face of K generated by $\{\omega_{e_j \otimes f_j} \mid [e_j] = [e_i]\}$, and is affinely isomorphic to the state space of $\mathcal{B}(L_i)$.

We will show L_1, \dots, L_q are independent (i.e., that $L_1 + L_2 + \dots + L_q$ is a vector space direct sum). For $1 \leq i \leq q$ let $e_i \otimes g_i$ be a nonzero vector in L_i . For $i \neq j$, g_i and g_j are linear combinations of disjoint subsets of f_1, f_2, \dots, f_p , so by independence of f_1, f_2, \dots, f_p , the subset $\{g_1, \dots, g_q\}$ is independent. Thus by Lemma 2, $\{e_1 \otimes g_1, \dots, e_p \otimes g_p\}$ is independent, and hence the subspaces L_1, \dots, L_q are independent. Hence by Lemma 1, the convex hull of the faces F_i is a direct convex sum of those faces.

Finally, we need to show that this convex hull coincides with the face F of S . Extreme points of F are extreme points of S , so are pure product states. Suppose that $\omega_{x \otimes y}$ is a pure product state in F . Then $\omega_{x \otimes y}$ is in the face of K generated by $\{\omega_{e_i \otimes f_i} \mid 1 \leq i \leq p\}$, so $x \otimes y$ is in $\text{span}\{e_i \otimes f_i \mid 1 \leq i \leq p\}$. By Lemma 2, $[x] = [e_j]$ for some j , and $y \in \text{span}\{y_i \mid [e_i] = [e_j]\}$. Hence $\omega_{x \otimes y} \in F_j$. Thus each extreme point of F is in some F_j , so F is contained in the convex hull of $\{F_i \mid 1 \leq i \leq q\}$. Evidently F contains every F_j , so this convex hull equals F . \square

In Theorem 4 we showed that the face F is the direct convex sum of faces that are affinely isomorphic to state spaces of full matrix algebras. Convex sets of this type were studied by Vershik (in both finite and infinite dimensions), who called them *block simplexes* [21]. Other examples are provided by state spaces of any finite dimensional C^* -algebra. Our Theorem 4 provides new examples of such block simplexes.

Corollary 5. *Let e_1, e_2, \dots, e_p and f_1, f_2, \dots, f_p be unit vectors in \mathbb{C}^m and \mathbb{C}^n respectively. We assume that $[e_i] \neq [e_j]$ for $i \neq j$, and that f_1, f_2, \dots, f_p are linearly independent. If $\lambda_1, \dots, \lambda_k$ are nonnegative numbers with sum 1, then the separable state $\omega = \sum_i \lambda_i \omega_{e_i \otimes f_i}$ has a unique representation as a convex combination of pure product states.*

Proof. Suppose ω equals the convex combination $\sum_j \gamma_j \tau_j$ where each τ_j is a pure product state. Then each τ_j is in the face F of S generated by ω . By Theorem 4, F is a simplex, and the extreme points of F are all of the form $\omega_{e_i \otimes f_i}$. Since each τ_j is a vector state, it is a pure state as well, so each state τ_j must be an extreme point of F , and thus must equal some $\omega_{e_i \otimes f_i}$. Uniqueness of the representation of ω follows from the uniqueness of convex decompositions into extreme points of a (finite dimensional) simplex. \square

Definition. A separable state ω has *length* k if ω can be expressed as a convex combination of k pure product states and admits no decomposition into fewer than k pure product states. We denote by S_k the set of separable states of length at most k .

Definition. A separable state ω has a *unique decomposition* if it can be written as a convex combination of pure product states in just one way

By the above result, roughly speaking decompositions of separable states on $\mathcal{B}(\mathbb{C}^m \otimes \mathbb{C}^n)$ of length $\leq \max(m, n)$ generically are unique. Here's a more precise statement.

Let $k \leq \max(m, n)$, and let V_k be the set of states ω admitting a convex decomposition $\omega = \sum_{i=1}^k \lambda_i \omega_{e_i \otimes f_i}$, where e_1, \dots, e_k and f_1, \dots, f_k are unit vectors in \mathbb{C}^m and \mathbb{C}^n respectively, $0 < \lambda_i$ for $1 \leq i \leq k$, $[e_1], \dots, [e_k]$ are distinct, and $\{f_1, \dots, f_k\}$ is linearly independent.

Theorem 6. *Let $m, n > 1$. For a given $k \leq \max(m, n)$, the states in V_k have length k , and have unique decompositions. The set V_k is open and dense in the set S_k of separable states of length at most k .*

Proof. Without loss of generality, we may assume $m \leq n$. By Corollary 5, each $\omega \in V_k$ admits a unique representation as a convex combination of pure product states, and each state in V_k has length k . We will show that V_k is open and dense in S_k .

To prove density, let $\omega \in S_k$ have a convex decomposition $\omega = \sum_{i=1}^k \lambda_i \omega_{x_i \otimes y_i}$. By slightly perturbing the coefficients λ_i if necessary, we may assume that $\lambda_i > 0$ for all i .

Given $\epsilon > 0$, by perturbing each x_i and y_i if necessary, we can find a second convex combination of pure product states $\omega' = \sum_{i=1}^k \lambda_i \omega_{e_i \otimes f_i}$ with $\|\omega - \omega'\| < \epsilon$, with $[e_1], \dots, [e_k]$ distinct, and with $\{f_1, \dots, f_k\}$ independent. (Indeed, to achieve independence we may append unit vectors y_{k+1}, \dots, y_n to the vectors y_1, \dots, y_k to give the subset $\{y_1, y_2, \dots, y_n\}$ of \mathbb{C}^n , and by small perturbations arrange that the determinant of the matrix with columns y_1, \dots, y_n is nonzero.) Thus V_k is dense in S_k .

Let $I_0 = \{(\lambda_1, \lambda_2, \dots, \lambda_k) \in [0, 1]^k \mid \sum_i \lambda_i = 1\}$. Let U_m be the unit sphere of \mathbb{C}^m and U_n the unit sphere of \mathbb{C}^n . Let $X = I_0 \times (U_m)^k \times (U_n)^k$. Define $\psi : X \rightarrow S$ by

$$\psi((\lambda_1, \dots, \lambda_k), (x_1, \dots, x_k), (y_1, \dots, y_k)) = \sum_i \lambda_i \omega_{x_i \otimes y_i}.$$

Note that ψ is continuous, that X is compact with respect to the product topology, and that $\psi(X) = S_k$.

Now let X_0 be the set $\{((\lambda_1, \dots, \lambda_k), (x_1, \dots, x_k), (y_1, \dots, y_k))\}$ of members of X of such that $[x_1], \dots, [x_k]$ are distinct, such that $\{y_1, \dots, y_k\}$ is linearly independent, and such that $\lambda_i > 0$ for $1 \leq i \leq k$. By lower semicontinuity of the rank of a matrix whose columns are y_1, \dots, y_k , the set of elements $((\lambda_1, \dots, \lambda_k), (x_1, \dots, x_k), (y_1, \dots, y_k))$ of X with $\{y_1, \dots, y_k\}$ linearly independent is open in X , so it is clear that X_0 is an open subset of X . By construction, $\psi(X_0) = V_k$. Since X_0 is open in X , then $X \setminus X_0$ is closed and hence compact. Since ψ maps $X \setminus X_0$ onto $S_k \setminus \psi(X_0)$, then the latter is closed, so $V_k = \psi(X_0)$ is open in S_k . \square

As remarked in the introduction, the sets V_k are open and dense in the relative topology on S_k , but are not open or dense in S or K if

$mn > 1$. To see this recall that a point σ in a convex set C is an algebraic interior point if for every point ρ in C there is a point τ in C such that σ lies on the open line segment from ρ to τ . Clearly for every algebraic interior point σ of S and every pure product state ρ , there is a convex decomposition of σ that includes ρ with positive weight. Since there are infinitely many pure product states, there are infinitely many convex decompositions for every algebraic interior point of S .

Every nonempty subset which is open in S contains an algebraic interior point of S ([19, pp. 88-91]), so contains points with nonunique decompositions. Thus V_k is not open in S or K . It is not dense in S or K , since for any m, n there exists $r > 0$ such that all states σ within a distance r from the normalized tracial state are separable, cf. [25, Thm. 1]. Every such state σ is an algebraic interior point of S , and so fails to have a unique decomposition.

Observe that Theorem 6 implies that V_k is also open and dense in the set of separable states of length equal to k .

4. DESCRIPTION OF CONVEX DECOMPOSITIONS

Let e_1, e_2, \dots, e_p and f_1, f_2, \dots, f_p be unit vectors in \mathbb{C}^m and \mathbb{C}^n respectively, with f_1, \dots, f_p linearly independent. Suppose ω is a convex combination of $\{\omega_{e_i \otimes f_i} \mid 1 \leq i \leq p\}$. In this section, we will describe all convex decompositions of ω into pure product states.

Let $\omega = \sum_i \lambda_i \omega_i$ be any convex decomposition of ω into pure product states. Then following the notation of Theorem 4, each ω_i is in $\text{face}_S(\omega) \subset F$. Since each ω_i is an extreme point of S , and F is the direct convex sum of the faces F_i , then each ω_i must be in some F_k . If we define $\gamma_k = \sum_{\{i \mid \omega_i \in F_k\}} \lambda_i$ and $\sigma_k = \gamma_k^{-1} \sum_{\{i \mid \omega_i \in F_k\}} \lambda_i \omega_i$, then ω has the convex decomposition

$$(6) \quad \omega = \sum_k \gamma_k \sigma_k \text{ with } \sigma_k \in F_k \text{ for each } k.$$

Since F is the direct convex sum of the F_k , the decomposition of ω in (6) is unique.

All possible convex decompositions of ω into pure product states can be found by starting with the unique decomposition $\omega = \sum_k \gamma_k \sigma_k$ with $\sigma_k \in F_k$, and then decomposing each σ_k into pure states. (Every state in F_k is separable, so pure states are pure product states). Since F_k is affinely isomorphic to the state space of $\mathcal{B}(L_k)$, unless each σ_k is itself a pure state, this can be done in many ways, as we discussed in the introduction. The possibilities have been described in [24, 18, 10].

A decomposition of a separable state ω as a convex combination of pure product states can be interpreted as a representation of ω as the

barycenter of a probability measure on the extreme points of S . With this interpretation the statement above can be rephrased in terms of the concept of dilation of measures (as defined e.g. in [1, p. 25]). If ω is given as above, then the probability measures on pure product states that represent ω are precisely those which are dilations of the uniquely determined probability measure $\mu = \sum_k \gamma_k \mu_k$ obtained from (6) with $\mu_k = \delta_{\sigma_k}$.

5. AFFINE AUTOMORPHISMS OF THE SPACE S OF SEPARABLE STATES

Notation. Fix m, n . We denote the state space of $\mathcal{B}(\mathbb{C}^m)$ by K_m , the state space of $\mathcal{B}(\mathbb{C}^n)$ by K_n , and the state space of $\mathcal{B}(\mathbb{C}^m \otimes \mathbb{C}^n)$ by K or $K_{m,n}$. The convex set of separable states in K is denoted by S or $S_{m,n}$. We will sometimes deal with a second algebra $\mathcal{B}(\mathbb{C}^{m'} \otimes \mathbb{C}^{n'})$, whose state space and separable state spaces we will denote by K' or S' respectively.

From Theorem 4, the face of S generated by two distinct pure product states $\omega_1 \otimes \sigma_1$ and $\omega_2 \otimes \sigma_2$ is a line segment (if $\omega_1 \neq \omega_2$ and $\sigma_1 \neq \sigma_2$) or is isomorphic to the state space of $\mathcal{B}(\mathbb{C}^2)$ and hence is a 3-ball (when $\omega_1 = \omega_2$ but $\sigma_1 \neq \sigma_2$, or when $\sigma_1 = \sigma_2$ but $\omega_1 \neq \omega_2$). (By a 3-ball we mean a convex set affinely isomorphic to the closed unit ball of \mathbb{R}^3 . The fact that the state space of $\mathcal{B}(\mathbb{C}^2)$ is a 3-ball can be found in many places, e.g., [2, Thm. 4.4].)

We define a relation R on the pure product states of K by $\rho R \tau$ if $\text{face}_S(\rho, \tau)$ is a 3-ball. By the remarks above, $(\omega_1 \otimes \sigma_1) R (\omega_2 \otimes \sigma_2)$ iff $(\omega_1 = \omega_2 \text{ but } \sigma_1 \neq \sigma_2)$ or $(\sigma_1 = \sigma_2 \text{ but } \omega_1 \neq \omega_2)$. Note that an affine isomorphism $\Phi : S \rightarrow S'$ will take faces of S to faces of S' , and will take 3-balls to 3-balls, so for pure product states ρ, τ we have $\rho R \tau$ iff $\Phi(\rho) R \Phi(\tau)$.

The idea of the following lemmas is to show that if $\Phi(\omega \otimes \sigma) = \phi(\omega, \sigma) \otimes \psi(\omega, \sigma)$, then ϕ depends only on the first argument and ψ depends only on the second argument, or possibly vice versa. Although we are interested in affine automorphisms of a single space of separable states, it will be easier to establish the needed lemmas in the context of affine isomorphisms from S to S' .

We use the notation $\partial_e C$ for the set of extreme points of a convex set C . For example, $\partial_e K$ is the set of pure states on $\mathcal{B}(\mathbb{C}^m \otimes \mathbb{C}^n)$.

Lemma 7. *Let $\Phi : S_{m,n} \rightarrow S_{m',n'}$ be an affine isomorphism. Let ω_1, ω_2 be distinct pure states in K_m and σ_1, σ_2 distinct pure states in K_n .*

Then the following four equations cannot hold simultaneously.

$$\begin{aligned}\Phi(\omega_1 \otimes \sigma_1) &= \rho_1 \otimes \tau_1 \\ \Phi(\omega_1 \otimes \sigma_2) &= \rho_1 \otimes \tau_2 \\ \Phi(\omega_2 \otimes \sigma_1) &= \rho_2 \otimes \tau_3 \\ \Phi(\omega_2 \otimes \sigma_2) &= \rho_3 \otimes \tau_3\end{aligned}$$

(7)

for $\rho_1, \rho_2, \rho_3 \in \partial_e K_{m'}$ and $\tau_1, \tau_2, \tau_3 \in \partial_e K_{n'}$.

Proof. We assume for contradiction that all four equations hold. Since $(\omega_1 \otimes \sigma_1) \text{ R } (\omega_2 \otimes \sigma_1)$, then $(\rho_1 \otimes \tau_1) \text{ R } (\rho_2 \otimes \tau_3)$. Hence

$$(8) \quad \rho_1 = \rho_2 \text{ or } \tau_1 = \tau_3.$$

Similarly $(\omega_1 \otimes \sigma_2) \text{ R } (\omega_2 \otimes \sigma_2)$, so $(\rho_1 \otimes \tau_2) \text{ R } (\rho_3 \otimes \tau_3)$. Hence

$$(9) \quad \rho_1 = \rho_3 \text{ or } \tau_2 = \tau_3.$$

Since we are assuming that $\omega_1 \neq \omega_2$ and $\sigma_1 \neq \sigma_2$, the four states $\{\omega_i \otimes \sigma_j \mid 1 \leq i, j \leq 2\}$ are distinct, so the four states on the right side of (7) must be distinct. Combining (8) and (9) gives four possibilities, each contradicting the fact that the states on the right side of (7) are distinct. Indeed:

$$\begin{aligned}(\rho_1 = \rho_2 \text{ and } \rho_1 = \rho_3) &\implies \rho_2 \otimes \tau_3 = \rho_3 \otimes \tau_3 \\ (\rho_1 = \rho_2 \text{ and } \tau_2 = \tau_3) &\implies \rho_1 \otimes \tau_2 = \rho_2 \otimes \tau_3 \\ (\tau_1 = \tau_3 \text{ and } \rho_1 = \rho_3) &\implies \rho_1 \otimes \tau_1 = \rho_3 \otimes \tau_3 \\ (\tau_1 = \tau_3 \text{ and } \tau_2 = \tau_3) &\implies \rho_1 \otimes \tau_1 = \rho_1 \otimes \tau_2.\end{aligned}$$

We conclude that the four equations in (7) cannot hold simultaneously. \square

Definition. Recall that we identify $\mathcal{B}(\mathbb{C}^m \otimes \mathbb{C}^n)$ with $\mathcal{B}(\mathbb{C}^m) \otimes \mathcal{B}(\mathbb{C}^n)$. The *swap isomorphism* $(\alpha_{m,n})_* : \mathcal{B}(\mathbb{C}^n \otimes \mathbb{C}^m) \rightarrow \mathcal{B}(\mathbb{C}^m \otimes \mathbb{C}^n)$ is the *-isomorphism that satisfies $(\alpha_{m,n})_*(A \otimes B) = B \otimes A$. If operators in $\mathcal{B}(\mathbb{C}^m \otimes \mathbb{C}^n)$ are identified with matrices, the swap isomorphism is the same as the “canonical shuffle” discussed in [13, Chapter 8]. The dual map $\alpha_{m,n}$ is an affine isomorphism from the state space of $\mathcal{B}(\mathbb{C}^m \otimes \mathbb{C}^n)$ to the state space of $\mathcal{B}(\mathbb{C}^n \otimes \mathbb{C}^m)$, with $\alpha_{m,n}(\omega \otimes \sigma) = \sigma \otimes \omega$. This restricts to an affine isomorphism from $S_{m,n}$ to $S_{n,m}$, which we also refer to as the swap isomorphism. If $m = n$, then $(\alpha_{m,m})_*$ is a *-automorphism of $\mathcal{B}(\mathbb{C}^m \otimes \mathbb{C}^m)$, $\alpha_{m,m}$ is an affine automorphism of the state space K , and restricts to an affine automorphism of the space S of separable states.

Lemma 8. *Let $\Phi : S_{m,n} \rightarrow S_{m',n'}$ be an affine isomorphism. At least one of the following two possibilities occurs:*

- (i) *For every $\omega \in \partial_e K_m$ there exists $\rho \in \partial_e K_{m'}$ such that $\Phi(\omega \otimes K_n) = \rho \otimes K_{n'}$, and for every $\sigma \in \partial_e K_n$ there exists $\tau \in \partial_e K_{n'}$ such that $\Phi(K_m \otimes \sigma) = K_{m'} \otimes \tau$.*
- (ii) *For each $\omega \in \partial_e K_m$ there exists $\tau \in \partial_e K_{n'}$ such that $\Phi(\omega \otimes K_n) = K_{m'} \otimes \tau$, and for every $\sigma \in \partial_e K_n$ there exists $\rho \in \partial_e K_{m'}$ such that $\Phi(K_m \otimes \sigma) = \rho \otimes K_{n'}$.*

If (i) occurs, then $m = m'$ and $n = n'$. If (ii) occurs, then $m = n'$ and $n = m'$.

Proof. For fixed $\omega \in \partial_e K_m$ and distinct $\sigma_1, \sigma_2 \in \partial_e K_n$ we have $(\omega \otimes \sigma_1) \text{ R } (\omega \otimes \sigma_2)$, so $\Phi(\omega \otimes \sigma_1) \text{ R } \Phi(\omega \otimes \sigma_2)$. Thus either there exist $\rho_1 \in \partial_e K_{m'}$ and distinct $\tau_1, \tau_2 \in \partial_e K_{n'}$ such that

$$(10) \quad \Phi(\omega \otimes \sigma_i) = \rho_1 \otimes \tau_i \text{ for } i = 1, 2,$$

or there exist distinct $\rho_1, \rho_2 \in \partial_e K_{m'}$ and $\tau_3 \in \partial_e K_{n'}$ such that

$$(11) \quad \Phi(\omega \otimes \sigma_i) = \rho_i \otimes \tau_3 \text{ for } i = 1, 2.$$

We will show that (10) implies (i), and (11) implies (ii).

Suppose that (10) holds. Let $\sigma \in \partial_e K_n$ with $\sigma \neq \sigma_1$ and $\sigma \neq \sigma_2$, and let $\Phi(\omega \otimes \sigma) = \rho \otimes \tau$. Since $(\omega \otimes \sigma) \text{ R } (\omega \otimes \sigma_i)$ for $i = 1, 2$, then $(\rho \otimes \tau) \text{ R } (\rho_1 \otimes \tau_i)$ for $i = 1, 2$. Hence $(\rho = \rho_1 \text{ or } \tau = \tau_1)$ and $(\rho = \rho_1 \text{ or } \tau = \tau_2)$. Since $\tau_1 \neq \tau_2$, then $\rho = \rho_1$. It follows that $\Phi(\omega \otimes K_n) \subset \rho_1 \otimes K_{n'}$. Thus

$$(12) \quad \Phi(\omega \otimes \sigma_i) = \rho_1 \otimes \tau_i \text{ for } i = 1, 2 \implies \Phi(\omega \otimes K_n) \subset \rho_1 \otimes K_{n'}.$$

Now (10) also implies

$$(13) \quad \Phi^{-1}(\rho_1 \otimes \tau_i) = \omega \otimes \sigma_i \text{ for } i = 1, 2.$$

If (10) holds (and hence also (13)), then applying the implication (12) to (13) with Φ^{-1} in place of Φ shows $\Phi^{-1}(\rho_1 \otimes K_{n'}) \subset \omega \otimes K_n$, so by (12) equality holds. Hence we have shown

$$(14) \quad \Phi(\omega \otimes \sigma_i) = \rho_1 \otimes \tau_i \text{ for } i = 1, 2 \implies \Phi(\omega \otimes K_n) = \rho_1 \otimes K_{n'}.$$

Now suppose instead that (11) holds. Let $\alpha_{m',n'}$ be the swap affine isomorphism defined above, so that $\alpha_{m',n'} : S_{m',n'} \rightarrow S_{n',m'}$. Then

$$(15) \quad (\alpha_{m',n'} \circ \Phi)(\omega \otimes \sigma_i) = \alpha_{m',n'}(\rho_i \otimes \tau_3) = \tau_3 \otimes \rho_i \text{ for } i = 1, 2.$$

By the implication (14) applied to $\alpha_{m',n'} \circ \Phi$ we conclude that

$$(\alpha_{m',n'} \circ \Phi)(\omega \otimes K_n) = \tau_3 \otimes K_{m'},$$

so

$$\Phi(\omega \otimes K_n) = \alpha_{m',n'}^{-1}(\tau_3 \otimes K_{m'}) = K_{m'} \otimes \tau_3.$$

Thus we have proven the implication

$$(16) \quad \Phi(\omega \otimes \sigma_i) = \rho_i \otimes \tau_3 \text{ for } i = 1, 2 \implies \Phi(\omega \otimes K_n) = K_{m'} \otimes \tau_3.$$

By Lemma 7 and the implications (14) and (16), either (10) must hold for all $\omega \in \partial_e K_m$ or (11) must hold for all $\omega \in \partial_e K_m$. We conclude that either

$$(17) \quad \forall \omega \in \partial_e K_m \quad \exists \rho \in \partial_e K_{m'} \text{ such that } \Phi(\omega \otimes K_n) = \rho \otimes K_{n'}$$

or

$$(18) \quad \forall \omega \in \partial_e K_m \quad \exists \tau \in \partial_e K_{n'} \text{ such that } \Phi(\omega \otimes K_n) = K_{m'} \otimes \tau.$$

Similarly, either

$$(19) \quad \forall \sigma \in \partial_e K_n \quad \exists \tau' \in \partial_e K_{n'} \text{ such that } \Phi(K_m \otimes \sigma) = K_{m'} \otimes \tau'$$

or

$$(20) \quad \forall \sigma \in \partial_e K_n \quad \exists \rho' \in \partial_e K_{m'} \text{ such that } \Phi(K_m \otimes \sigma) = \rho' \otimes K_{n'}.$$

Suppose that (17) and (20) both held. For $\omega \in K_m$ and $\sigma \in K_n$ note that $\omega \otimes \sigma$ is in both $\omega \otimes K_n$ and $K_m \otimes \sigma$, so $\rho \otimes K_{n'}$ and $\rho' \otimes K_{n'}$ are not disjoint. This implies $\rho = \rho'$, so $\Phi(\omega \otimes K_n) = \Phi(K_m \otimes \sigma)$. Since Φ is bijective, $\omega \otimes K_n = K_m \otimes \sigma$ follows. This is possible only if $m = n = 1$. If $m = n = 1$, then all of (17), (18), (19), (20) hold. Similarly if (18) and (19) both held then $m = n = 1$ is again forced. Thus the possibilities are that (17) and (19) both hold (which is the same as statement (i) of the lemma), or that (18) and (20) hold (equivalent to (ii)), or that $m = n = 1$, in which case both (i) and (ii) hold.

Finally, since the affine dimensions of K_p and K_q are different when $p \neq q$, the statement in the last sentence of the lemma follows. \square

If $\psi_1 : K_m \rightarrow K_m$ and $\psi_2 : K_n \rightarrow K_n$ are affine automorphisms, then we can extend each to linear maps on the linear span, and form the tensor product $\psi_1 \otimes \psi_2$. This will be bijective, but not necessarily positive. (A well known example of this phenomenon occurs when ψ_1 is the identity map and ψ_2 is the transpose map.) However, ψ_1 and ψ_2 will map pure states to pure states, and hence $\psi_1 \otimes \psi_2$ will map pure product states to pure product states. Thus $\psi_1 \otimes \psi_2$ will map S onto S , and hence will be an affine automorphism of S . We will now see that all affine automorphisms of S are either such a tensor product of automorphisms or such a tensor product composed with the swap automorphism.

Theorem 9. *If $m \neq n$, and $\Phi : S \rightarrow S$ is an affine automorphism, then there exist unique affine automorphisms $\psi_1 : K_m \rightarrow K_m$ and $\psi_2 : K_n \rightarrow K_n$ such that $\Phi = \psi_1 \otimes \psi_2$. If $m = n$ then either we can write $\Phi = (\psi_1 \otimes \psi_2)$ or $\Phi = \alpha_{m,m} \circ (\psi_1 \otimes \psi_2)$, where ψ_1, ψ_2 are again unique affine automorphisms of K_m and K_n respectively, and $\alpha_{m,m} : S \rightarrow S$ is the swap automorphism.*

Proof. We apply Lemma 8. For each $\omega \in \partial_e K_m$ and $\sigma \in \partial_e K_n$, define $\phi_\sigma : K_m \rightarrow K_m$ and $\psi_\omega : K_n \rightarrow K_n$ by

$$\Phi(\omega \otimes \sigma) = \phi_\sigma(\omega) \otimes \psi_\omega(\sigma).$$

Suppose first that case (i) of Lemma 8 occurs. Then $\psi_\sigma(\omega)$ is independent of σ and $\psi_\omega(\sigma)$ is independent of ω . Therefore there are functions $\psi_1 : K_m \rightarrow K_m$ and $\psi_2 : K_n \rightarrow K_n$ such that

$$\Phi(\omega \otimes \sigma) = \psi_1(\omega) \otimes \psi_2(\sigma).$$

Since Φ is bijective and affine, so are ψ_1 and ψ_2 .

Suppose instead that case (ii) of Lemma 8 occurs. Then $m = n$. If we define $\Phi' = \alpha_{m,m} \circ \Phi$, then $\Phi' : S \rightarrow S$ satisfies case (i) of Lemma 8. Then from the first paragraph we can choose affine automorphisms $\psi_1 : K_m \rightarrow K_m$ and $\psi_2 : K_n \rightarrow K_n$ such that $\Phi' = \psi_1 \otimes \psi_2$. Since $\alpha_{m,m}^2$ is the identity map, then $\Phi = \alpha_{m,m} \circ (\psi_1 \otimes \psi_2)$. \square

We review some well known facts about affine automorphisms of state spaces and maps on the underlying algebra. Let ψ be an affine automorphism of K_m . Then ψ extends uniquely to a linear map on the linear span of K_m , which we also denote by ψ , and this map is the dual of a unique linear map ψ_* on $\mathcal{B}(\mathbb{C}^m)$. By a result of Kadison [9] ψ_* will be a $*$ -isomorphism or a $*$ -anti-isomorphism. (Since the restriction of an affine automorphism to pure states preserves transition probabilities, this also follows from Wigner's theorem [23]). The map ψ_* will be a $*$ -isomorphism iff ψ_* is completely positive, which is equivalent to ψ being completely positive. If ψ_* is a $*$ -isomorphism, then ψ_* is implemented by a unitary, i.e., there is a unitary $U \in \mathcal{B}(\mathbb{C}^m)$ such that $\psi_*(A) = UAU^*$.

If ψ_* is a $*$ -anti-isomorphism, then the composition of ψ_* with the transpose map (in either order) gives a $*$ -isomorphism, and the map ψ_* is completely copositive. It follows that an affine automorphism ψ of K_m is either completely positive or completely copositive, and ψ is completely positive iff ψ^{-1} is completely positive. If t denotes the transpose map on $\mathcal{B}(\mathbb{C}^m)$ or $\mathcal{B}(\mathbb{C}^n)$, then t is positive but $t \otimes id$ and $id \otimes t$ are not positive on $\mathcal{B}(\mathbb{C}^m) \otimes \mathcal{B}(\mathbb{C}^n)$ if $m, n > 1$. Background can be found in [2, Chapters 4, 5].

Recall that a *local unitary* in $\mathcal{B}(\mathbb{C}^m \otimes \mathbb{C}^n)$ is a tensor product $U_1 \otimes U_2$ of unitaries.

Theorem 10. *Every affine automorphism of the space S of separable states on $\mathcal{B}(\mathbb{C}^m \otimes \mathbb{C}^n)$ is the dual of conjugation by local unitaries, one of the two partial transpose maps, the swap map (if $m = n$), or a composition of these maps. An affine automorphism Φ of S extends uniquely to an affine automorphism of the full state space K iff it can be expressed as one of the compositions just mentioned with both or neither of the partial transpose maps involved.*

Proof. We note first that if $m = 1$ or $n = 1$, the result is clear, so we assume hereafter that $m \geq 2$ and $n \geq 2$.

We next show that if $\psi_1 : K_m \rightarrow K_m$ and $\psi_2 : K_n \rightarrow K_n$ are affine automorphisms, then $\Phi = \psi_1 \otimes \psi_2$ is an affine automorphism of K iff ψ_1 and ψ_2 are both completely positive or both completely copositive.

If ψ_1 and ψ_2 are completely positive, then $\Phi = \psi_1 \otimes \psi_2 = (id \otimes \psi_2) \circ (\psi_1 \otimes id)$ is positive; hence $\Phi(K) \subset K$. Furthermore, ψ_1^{-1} and ψ_2^{-1} will be completely positive, so Φ^{-1} is positive, and hence $\Phi(K) = K$. If ψ_1 and ψ_2 are completely copositive, then $(t \circ \psi_1) \otimes (t \circ \psi_2)$ is positive. Composing with $t \otimes t$ shows $\psi_1 \otimes \psi_2$ is positive and as above we conclude that $\Phi(K) = K$. On the other hand, if ψ_1 is completely positive and ψ_2 is completely copositive, then $\psi_1 \otimes (t \circ \psi_2)$ is positive, so $(id \otimes t) \circ (\psi_1 \otimes \psi_2)$ is positive. If $(\psi_1 \otimes \psi_2)(K) = K$, then $id \otimes t$ would be positive, a contradiction since $m, n \geq 2$. Thus in this case $\psi_1 \otimes \psi_2$ is not an affine automorphism of K .

If ψ_1 and ψ_2 are completely positive, then they are implemented by unitaries, so $\Phi = \psi_1 \otimes \psi_2$ is implemented by a local unitary. If both are completely copositive, then $t \circ \psi_1$ and $t \circ \psi_2$ are implemented by unitaries, so $(t \otimes t) \circ (\psi_1 \otimes \psi_2)$ is implemented by a local unitary. Then $\Phi = (t \otimes t) \circ (t \otimes t) \circ (\psi_1 \otimes \psi_2)$ is the composition of the transpose map on K and conjugation by local unitaries.

The first statement of the theorem now follows from Theorem 9. Uniqueness follows from the fact that the linear span of S contains K . \square

Definition. Let $\Phi : K \rightarrow K$ be an affine automorphism. We say Φ *preserves separability* if Φ takes separable states to separable states, i.e., if $\Phi(S) \subset S$. A state ω in K is *entangled* if ω is not separable. Φ *preserves entanglement* if Φ takes entangled states to entangled states.

Corollary 11. *Let $\Phi : K_{m,n} \rightarrow K_{m,n}$ be an affine automorphism. Then Φ preserves entanglement and separability iff Φ is a composition*

of maps of the types (i) conjugation by local unitaries, (ii) the transpose map, (iii) the swap automorphism (in the case that $m = n$).

Proof. If Φ preserves entanglement and separability, then Φ maps S into S and $K \setminus S$ into $K \setminus S$, which is equivalent to $\Phi(S) = S$. \square

Corollary 12. *If $\Phi_t : S \rightarrow S$ is a one-parameter group of affine automorphisms, then there are one-parameter groups of unitaries U_t and V_t such that $\Phi_t(\omega(A)) = \omega((U_t \otimes V_t)A(U_t^* \otimes V_t^*))$.*

Proof. For each t , factor $\Phi_t = \phi_t \otimes \psi_t$ or $\Phi_t = \alpha \circ (\phi_t \otimes \psi_t)$, where α is the swap automorphism. In the latter case,

$$\begin{aligned} \Phi_{2t} &= \Phi_t \circ \Phi_t = \alpha \circ (\phi_t \otimes \psi_t) \circ \alpha \circ (\phi_t \otimes \psi_t) \\ &= (\phi_t \otimes \psi_t) \circ (\phi_t \otimes \psi_t) = (\phi_t \circ \phi_t) \otimes (\psi_t \circ \psi_t). \end{aligned}$$

It follows that the swap automorphism is not needed for Φ_{2t} , and hence for Φ_t for any t . Uniqueness of the factorization $\Phi_t = \phi_t \otimes \psi_t$ shows that ϕ_t and ψ_t are also one parameter groups of affine automorphisms. By a result of Kadison [8], such automorphisms are implemented by one parameter groups of unitaries. \square

Corollary 13. *If $\Phi_t : K \rightarrow K$ is a one-parameter group of entanglement preserving affine automorphisms, then there are one-parameter groups of unitaries U_t and V_t such that $\Phi_t(\omega(A)) = \omega((U_t \otimes V_t)A(U_t^* \otimes V_t^*))$.*

Proof. Since Φ_t and $(\Phi_t)^{-1} = \Phi_{-t}$ preserve entanglement, then Φ_t maps S onto S , so this corollary follows from Corollary 12. \square

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